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Migdal transformations of O(2)-symmetric spin Hamiltonians

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Abstract. The existence of a massless phase in several different two-dimensional O(2)-symmetric spin models is investigated within the approximate Migdal/Kadanoff renormalisation transformation. The models investigated include the planar rotor model, the truncated model used by Nienhuis to derive exact results for O(2)-symmetric models, and the step model of Guttmann and Joyce. The truncated model is found to exhibit a 'fixed' line similar to that seen in the planar rotor model, but only for values of the coupling for which the model is unphysical. The step model is found to be always disordered, but a variant exhibits a fixed line. The significance of the results beyond the Migdal approximation is assessed.

1. Introduction

Some years ago, José *et al* (1977) used a simple Migdal–Kadanoff recursion relation (Migdal 1975, Kadanoff 1976) to study the two-dimensional planar rotor or classical xy model. This model exhibits a massless low-temperature phase terminated by a vortex unbinding (Kosterlitz–Thouless) transition (Kosterlitz and Thouless 1973, Kosterlitz 1974, José *et al* 1977). Within a renormalisation group transformation such a phase should correspond to a fixed line. While the Migdal approximation does not possess an exact fixed line (Migdal 1975, Wilson 1976, José *et al* 1977), José *et al* (1977) found numerically that successive iterations of the planar model for sufficiently low temperature appeared effectively to approach a non-trivial fixed point Hamiltonian closely approximating the Villain model (Villain 1975). Consequently, a semi-quantitative picture of the behaviour of the planar model could be obtained from the Migdal approximation despite the absence of a true fixed line.

The planar model is specified by the (reduced) Hamiltonian

$$-\beta H = K \sum_{\langle i,j\rangle} \cos(\theta_i - \theta_j), \qquad 0 \leqslant \theta_i \leqslant 2\pi, \tag{1.1}$$

where the sum is over all nearest-neighbour bonds of a two-dimensional lattice. Universality, however, implies that the critical behaviour of the model should not depend upon the particular interaction function (the cosine) appearing in (1.1); the

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crucial features being the two-dimensionality of the lattice, the short-ranged interaction, and, in particular, the global O(2)-symmetry of the model. Thus any two-dimensional Hamiltonian of the form

$$-\beta H = \sum_{\langle i,j\rangle} V(\theta_i - \theta_j), \tag{1.2}$$

where $V(\theta)$ has period 2π but can otherwise be chosen to facilitate the theoretical analysis, should exhibit quantitatively similar results to the planar model. To a certain extent, this idea is supported by the results of José *et al* (1977), in particular with regard to the Villain (1975) model defined by

$$\exp[V_v(\theta)] = \sum_{m=-\infty}^{\infty} \exp[-\tfrac{1}{2}K_v(\theta - 2\pi m)^2]. \tag{1.3}$$

Two other models have also received attention in recent years as 'simpler' models with which to explore the O(2)-symmetric universal class. These are the *step model* (Guttmann *et al* 1972) defined by

$$V(\theta) = \begin{cases} K & \text{for } |\theta| \leq \tfrac{1}{2}\pi, \\ -K & \text{for } \tfrac{1}{2}\pi < |\theta| \leq \pi, \end{cases} \qquad V(\theta) = V(\theta + 2\pi), \tag{1.4}$$

and what we shall call the *truncated model*† (Domany *et al* 1981) defined by

$$\exp[V(\theta)] = 1 + x \cos\theta. \tag{1.5}$$

Both these models were motivated, in part, by a desire to simplify the derivation of high-temperature series expansions for O(2) models. The truncated model has, however, more recently been used by Nienhuis (1982) to derive *exact* results for the critical temperature and thermal exponent of O(n) models on the hexagonal lattice for $-2 \leq n \leq 2$.

The O(n) analogue of (1.5) is defined by

$$\exp[V(\boldsymbol{s}, \boldsymbol{s}')] = 1 + x\boldsymbol{s} \cdot \boldsymbol{s}', \tag{1.6}$$

where $\boldsymbol{s}$ and $\boldsymbol{s}'$ are n-component unit vectors. Nienhuis actually normalises so that $|\boldsymbol{s}| = \sqrt{n}$. Hence his result (see also Domany *et al* 1981) for the critical temperature corresponds to $x_c = \sqrt{2}$, which is unphysical‡ in the sense that $\exp[V(\theta)]$ is not strictly positive for all θ. Whether this is a serious deficiency is unclear. Nor does Nienhuis's method of solution reveal whether the transition at x_c is to a massless phase. Here we explore these questions by considering the effect of a Migdal transformation on (1.5). While our results are derived for the square lattice rather than the hexagonal, we find that (1.5) exhibits no fixed line for physical values of x ($0 \leq x \leq 1$), but for $x \geq 1.5$ the model does iterate to an apparent fixed line.

The step model has also been extensively explored by series methods in both two and three dimensions (Guttmann and Joyce 1973, Guttmann and Nymeyer 1978). In three dimensions, estimates of the susceptibility exponent agree with that of the planar model, but in two dimensions the latest work (Guttmann and Nymeyer 1978) finds

† This model should not be confused with the truncation of the quantum Hamiltonian analogue of the planar model introduced by Luther and Scalapino (1977), and subsequently studied by Barber and Richardson (1981), Richardson and Hamber (1981) and den Nijs (1982). While the Euclidean equivalent of this truncated quantum Hamiltonian is unknown, it is *not* (1.5), despite a claim (Richardson and Hamber 1981) to this effect.

‡ Nienhuis's result is similarly unphysical for all $2 \geq n \geq n_c \simeq 1.6$.

no evidence of either a conventional algebraic singularity or the essential singularity expected at a Kosterlitz–Thouless transition. This conclusion is supported by the results reported here, no evidence of a fixed line being seen under Migdal transformations of (1.4). A 'fixed line' can, however, be recovered if the step in (1.4) is moved from $\pi/2$ to $\pi\delta$ with $\delta < \frac{1}{2}$.

The paper is arranged as follows. In § 2, we derive the Migdal–Kadanoff recursion relations for an O(2) Hamiltonian and discuss the role that the Villain model plays as an 'almost' fixed line. Our derivation differs from that given by José *et al* (1977) in that we define a bond-moving scheme so as to yield isotropic recursion relations for finite spatial rescaling. This facilitates the numerical studies reported in § 3. The paper closes with an overall summary in § 4, in which we try to assess the significance of our results beyond the context of the Migdal approximation.

2. Migdal recursion relations for O(2) Hamiltonians

2.1. *Derivation*

Consider an O(2)-symmetric Hamiltonian

$$-\beta H = \sum_{\langle i,j \rangle} V(\theta_i - \theta_j) \tag{2.1}$$

defined, for convenience, on a square lattice. The potential $V(\theta)$ will be assumed to satisfy

$$\text{(i)}\ \ V(\theta) = V(-\theta), \qquad \text{(ii)}\ \ V(\theta) = V(\theta + 2\pi), \qquad \text{(iii)}\ \ V(0) = 0, \tag{2.2}$$

with $V(\theta)$ continuous at $\theta = 0$. Define a Migdal–Kadanoff transformation by bond moving as shown in figure 1. Clearly, the spatial rescaling factor $b = 2$ and the renormalised potential is given by

$$A \exp[\bar{V}(\theta)] = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \exp[2V(\phi) + 2V(\theta - \phi)], \tag{2.3}$$

where A is determined so that $\bar{V}(0) = 0$. It is more convenient to work with the Boltzmann factor

$$f(\theta) = \exp[V(\theta)], \tag{2.4}$$

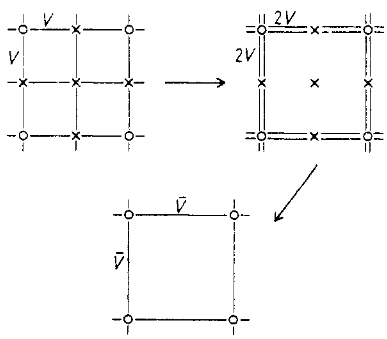

Figure 1. Isotropic bond moving scheme to define Migdal transformations. Sites marked × are integrated out in the transformation.

normalised so that $f(0)=1$. In terms of f, the recursion relation (2.3) then reads

$$\bar{f}(\theta)=\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}f^2(\phi)f^2(\theta-\phi)\Big/\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}f^4(\phi). \tag{2.5}$$

As mentioned earlier, the bond-moving scheme shown in figure 1 differs from that used by José *et al* (1977) and yields an isotropic recursion relation without the necessity of taking the limit $b\to 1$. However, the recursion relation (2.3) is identical to that obtained by José *et al* (1977) with $b=2$ for the interaction in the y direction, $V_y(\theta)$, that for the interaction in the x direction, $V_x(\theta)$, being different. Thus we expect and confirm in § 3 that the numerical results reported by José *et al* for the planar rotor model are reproduced by (2.5).

Following José *et al* (1977) we can easily implement (2.5) in the space of Fourier coefficients of $f(\theta)$. Let

$$f(\theta)=c_0+2\sum_{k=1}^{\infty}c_k\cos k\theta, \tag{2.6}$$

where

$$c_k=\pi^{-1}\int_0^{\pi}f(\theta)\cos k\theta\,\mathrm{d}\theta. \tag{2.7}$$

It is straightforward to show that

$$f^2(\theta)=b_0+2\sum_{l=1}^{\infty}b_l\cos l\theta, \tag{2.8}$$

where

$$b_l=\sum_{k=0}^{\infty}c_kc_{l-k}+2\sum_{k=1}^{\infty}c_kc_{l+k}. \tag{2.9}$$

Thus, by the Parseval relation,

$$\bar{f}(\theta)=\bar{c}_0+\sum_{k=1}^{\infty}\bar{c}_k\cos k\theta, \tag{2.10}$$

with

$$\bar{c}_k=b_k^2\Big/\left(b_0^2+2\sum_{l=1}^{\infty}b_l^2\right), \tag{2.11}$$

the numerator in (2.11) ensuring that $\bar{f}(0)=0$.

2.2. Numerical implementation

Provided the Fourier coefficients c_k approach zero sufficiently rapidly as k increases, (2.5) can be directly evaluated via (2.9) and (2.11). However, it is more convenient if the continuous O(2)-symmetry of the Hamiltonian (2.1) is replaced by a discrete Z_p-symmetry with large p. This is equivalent to sampling $f(\theta)$ at p equally spaced points in $[0, 2\pi]$. The required computation can then be performed by fast Fourier transform (FFT) techniques (see e.g. Hamming 1973).

Let

$$\theta_j=2\pi j/p, \qquad j=0,1,\ldots,p-1, \tag{2.12}$$

and

$$n = p/2 \tag{2.13}$$

where p is even. (The number of sample points may be taken as odd, in which case $n = (p+1)/2$ and the last term in (2.14) is multiplied by 2. However, FFT techniques are most easily implemented with p even.) We can then write (see Hamming 1973)

$$f^2(\theta) \simeq f_p^2(\theta) = a_0 + 2 \sum_{k=1}^{n-1} a_k \cos k\theta + a_n \cos n\theta, \tag{2.14}$$

with

$$a_k = (1/p) \sum_{j=0}^{p-1} f^2(\theta_j) \cos k\theta_j, \qquad k = 0, 1, \ldots, n. \tag{2.15}$$

Substituting (2.14) in (2.5) and replacing the integral by a sum over the θ_j's yields

$$\bar{f}(\theta_j) \simeq \bar{f}_p(\theta_j) = \frac{a_0^2 + 2\sum_{k=1}^{n-1} a_k^2 \cos k\theta_j + a_n^2 \cos n\theta_j}{a_0^2 + 2\sum_{n=1}^{n-1} a_k^2 + a_n^2}, \tag{2.16}$$

on the basis of which the transformation can be iterated.

Several comments on this method are appropriate. While (2.14) is exact at the sample points θ_j, (2.16) is not, the rate of convergence of $\bar{f}_p$ to $\bar{f}(\theta)$ being governed by the behaviour of c_k for large k. On the other hand, viewed as a transformation of the Z_p-symmetric system, (2.16) is an exact representation of the Migdal transformation. This replacement of a continuous symmetry by an appropriate discrete symmetry is common in numerical calculations on systems with continuous symmetries (see e.g. De Grand and Toussaint 1980). In addition, it is known that Z_p systems for $p > 5$ exhibit a fixed line (Elitzur *et al* 1979, Cardy 1980) and for large p behave qualitatively like the full O(2)-symmetric system except at very low temperatures ($k_B T \leqslant 4\pi^2/p^2$). This is borne out by our calculations; there is no significant change in our results for $p \geqslant 64$. Most of the calculations reported were performed with $p = 128$, these differing by less than 0.1% from those obtained with $p = 256$.

2.3. *The Villain model*

In our notation, the Villain model is most easily defined via its Fourier coefficients. Thus we write

$$f_v(\theta) = 1 + 2 \sum_{k=1}^{\infty} \exp(-k^2/2K) \cos k\theta, \tag{2.17}$$

omitting for the moment the normalisation to $f_v(0) = 1$. (The equivalence of this expression with (1.3) follows from the results of José *et al* (1977); see also Bellmann (1961, p 10).) Figure 2 shows this potential as a function of θ, together with one iteration under (2.5) for $K = 0.2$, 1.2 and 2.2. For $K = 1.2$ and 2.2, $f_v(\theta)$ is at least to graphical accuracy apparently a fixed-point solution of (2.5). To determine whether $f_v(\theta)$ is actually a fixed point of (2.5), we evaluate analytically the mapping defined by (2.5).

The function $f_v(\theta)$, defined by (2.17), is a theta function (see e.g. Whittaker and Watson 1978, Bellmann 1961). Explicitly

$$f_v(\theta) = \theta_3(\tfrac{1}{2}\theta, q)/\theta_3 \tag{2.18}$$

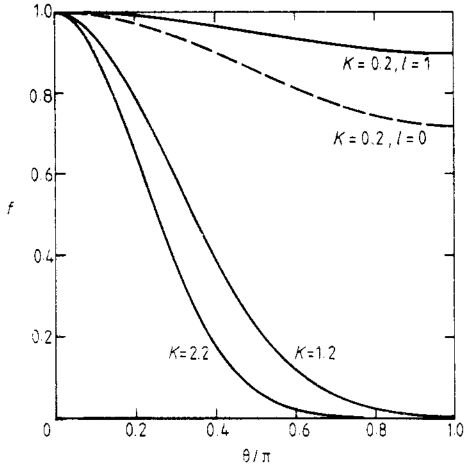

Figure 2. Villain model for $K = 0.2$ (- - -), 1.2 and 2.2 together with first iterates (——) under the Migdal transformation. For $K = 1.2$ and 2.2, the iterates are indistinguishable from the original curve.

where

$$q = e^{-K/2}, \tag{2.19}$$

$$\theta_3 = \theta_3(0, q), \tag{2.20}$$

$$\theta_3(z, q) = 1 + 2 \sum_{k=1}^{\infty} q^{k^2} \cos 2kz. \tag{2.21}$$

We shall also require one other theta function, namely

$$\theta_4(z, q) = 1 + 2 \sum_{k=1}^{\infty} (-)^k q^{k^2} \cos 2kz. \tag{2.22}$$

For brevity, the dependence on q will henceforth be suppressed and we shall also, as in (2.18), write $\theta_i = \theta_i(0) \equiv \theta_i(0, q)$. Both $\theta_3(z)$ and $\theta_4(z)$ are periodic functions of z with period π. In addition

$$\theta_3(z + \tfrac{1}{2}\pi) = \theta_4(z). \tag{2.23}$$

Iterating (2.18) under (2.5) yields the renormalised function

$$\bar{f}_v(\theta) = g(\tfrac{1}{2}\theta)/g(0), \tag{2.24}$$

where

$$g(z) = 2\int_0^{\pi} du\, \theta_3^2(u)\theta_3^2(z-u) = \int_0^{\pi} du[\theta_3^2(z-u)\theta_3^2(u) + \theta_4^2(u)\theta_4^2(z-u)], \tag{2.25}$$

the second equality following from (2.23) since the integrand is periodic with period π and the integration interval is a full period. To evaluate (2.25) we make use of the identity

$$\theta_3^2\theta_3(2x)\theta_3(2y) + \theta_4^2\theta_4(2x)\theta_4(2y) = \theta_3^2(x+y)\theta_3^2(x-y) + \theta_4^2(x+y)\theta_4^2(x-y), \tag{2.26}$$

which follows from the fundamental Jacobi formulae given on pp 467–8 of Whittaker and Watson (1978). Setting $x=\frac{1}{2}z$, $y=u-\frac{1}{2}z$ and substituting (2.26) in (2.25) gives

$$g(z)=\theta_3^2\theta_3(z)+\theta_4^2\theta_4(z) \tag{2.27}$$

since

$$\int_0^\pi \mathrm{d}u\,\theta_3(2u-z)=\int_0^\pi \mathrm{d}u\,\theta_4(2u-z)=1. \tag{2.28}$$

Thus

$$\bar{f}_{\mathrm{v}}(\theta)=[f_{\mathrm{v}}(\theta)+\gamma^2\Phi(\theta)]/(1+\gamma^3) \tag{2.29}$$

where

$$\gamma=\theta_4/\theta_3, \tag{2.30}$$

$$\Phi(\theta)=\theta_4(\tfrac{1}{2}\theta)/\theta_3. \tag{2.31}$$

Clearly $\bar{f}_{\mathrm{v}}(\theta)$ is not identical to $f_{\mathrm{v}}(\theta)$ except at $\theta=0$, which is determined by normalisation. To quantify the extent that $\bar{f}_{\mathrm{v}}$ and f differ, we introduce the infinite product representations of the theta functions (Whittaker and Watson 1978, p 470). Hence

$$\gamma=\prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n+1}}\right)^2, \tag{2.32}$$

$$\Phi(z)=\prod_{n=1}^{\infty}\frac{1-2q^{2n-1}\cos z+q^{4n-2}}{(1+q^{2n-1})^2}, \tag{2.33}$$

where $q=\mathrm{e}^{-K/2}$, which tends to unity as $K\to\infty$ $(T\to 0)$ so that the products are slowly convergent. However, an application of Jacobi's imaginary transformation (Whittaker and Watson 1978, p 476) yields

$$\gamma=2(q')^{1/4}\prod_{n=1}^{\infty}\frac{1+(q')^{2n}}{1+(q')^{2n-1}} \tag{2.34}$$

where $q'=\exp(-2\pi^2K)\to 0$ as $K\to\infty$. In fact, for all $0<K<\infty$,

$$0<\gamma<2\exp(-\pi^2K/2), \tag{2.35}$$

while from (2.28)

$$\Phi(z)<\Phi(\pi)=1. \tag{2.36}$$

Hence

$$\Delta f=\|\bar{f}_{\mathrm{v}}(\theta)-f_{\mathrm{v}}(\theta)\|_\infty=\max_{0\leqslant\theta\leqslant\pi}|\bar{f}_{\mathrm{v}}(\theta)-f_{\mathrm{v}}(\theta)|\leqslant\gamma^2\leqslant 4\exp(-\pi^2K) \tag{2.37}$$

for all K. This result together with (2.29) confirms, somewhat more rigorously and in a different norm, the conclusion of José *et al* (1977) that the Villain model is not a fixed line of the Migdal recursion relation (2.5) but fails only exponentially, the bound (2.36) being essentially zero for $K\gtrsim 1.0$ $(4\exp(-\pi^2)\sim 2\times 10^{-4})$.

As is evident from figure 3, the bound in (2.32) is a rather accurate estimate of the failure of the Villain model to be a fixed line for all $K\gtrsim 1$. Other measures, such as the root-mean-square error

$$\|\bar{f}_{\mathrm{v}}-f_{\mathrm{v}}\|_2=\left(\int_0^{2\pi}(\bar{f}_{\mathrm{v}}(\theta)-f_{\mathrm{v}}(\theta))^2\,\mathrm{d}\theta/2\pi\right)^{1/2} \tag{2.38}$$

behave similarly, $\|\bar{f}_{\mathrm{v}}-f_{\mathrm{v}}\|_2$ as expected being slightly smaller.

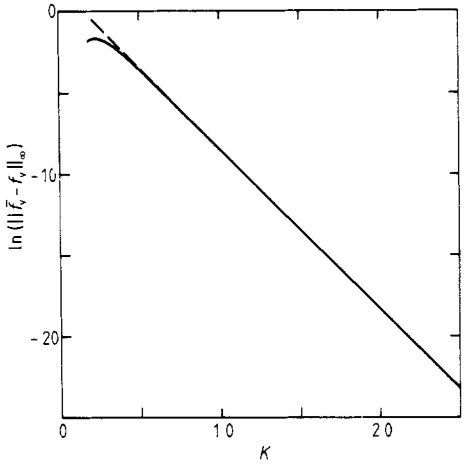

Figure 3. Natural logarithm of the maximum norm of f_v and $\bar{f}_v$ as a function of K. The broken curve is the theoretical bound (2.32).

3. Numerical results

In § 2 we saw that the Villain model fails only narrowly to be a fixed-line solution of the Migdal recursion relation (2.5). However, when analysed on a graphical scale, the Villain model appears as a fixed line of (2.5), corresponding to the expected massless low-temperature phase of an O(2)-symmetric model. In the light of this observation we now wish to use (2.5) to explore the behaviour of some other specific O(2)-models by enquiring whether, under iteration, these models converge in a similar way to a 'fixed line' for sufficiently low temperatures.

Specifically we consider:

(i) the planar model

$$f_p(\theta) = \exp[K(\cos\theta - 1)], \tag{3.1}$$

(ii) the truncated model

$$f_t(\theta) = (1 + x\cos\theta)/(1+x), \tag{3.2}$$

(iii) the step model

$$f_s(\theta) = \begin{cases} 1, & 0 \leq |\theta| \leq \delta\pi \\ e^{-2K}, & \delta\pi \leq |\theta| \leq \pi \end{cases}, \qquad f(\theta) = f(\theta + 2\pi), \tag{3.3}$$

(iv) the line model

$$f_l(\theta) = 1 - \beta|\theta|/\pi, \qquad f(\theta) = f(\theta + 2\pi). \tag{3.4}$$

Note that the step model has been generalised to allow an arbitrarily positioned 'step'. The physical significance of this modification will be discussed below. The line model is of interest both because of its simplicity and because its behaviour under the Migdal transformation is very similar to the truncated model.

3.1. *The planar model*

The behaviour of the planar model (3.1) under (2.5) has, of course, been discussed in some detail by José *et al* (1977). Here our interest is simply to review its behaviour so as to establish a benchmark against which to compare and contrast the behaviour of the other models.

Figure 4 shows the behaviour of the planar model under (2.5) for $K=0.5$ and $K=2.0$. Iterations for $K=0.5$ clearly drive the Boltzmann weight $f(\theta)$ to unity for all θ corresponding to the infinite-temperature fixed point, while for $K=2.0$, a non-trivial 'fixed function' is approached. The behaviour shown for $K=0.5$ is typical of that seen for $K\leqslant 1$, while $K=2.0$ is typical of the behaviour seen for $K\geqslant 1$ except that the limiting function depends upon K as expected of a fixed line. The limiting functions can be fitted to the Villain model (José *et al* 1977).

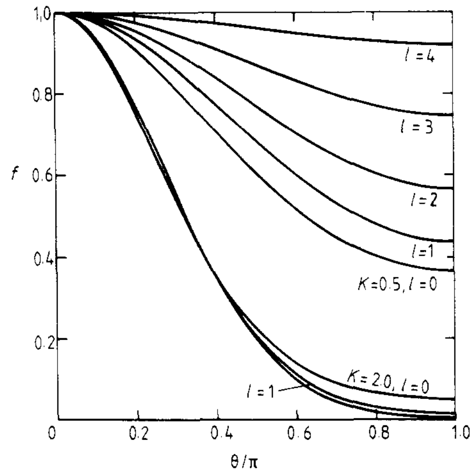

Figure 4. Successive Migdal iterations (labelled by l) of the planar rotor model for $K=0.5$ and $K=2.0$. Note the apparent convergence in the latter case to a 'fixed' function.

If we accept as a criterion of convergence three successive iterations differing by less than 10^{-4} in maximum norm, $\|f_k-f_{k+1}\|=\max_{0\leqslant\theta\leqslant\pi}|f_k(\theta)-f_{k+1}(\theta)|$, then the onset of the massless phase occurs for $K\simeq K_c\simeq 1.0$. While this estimate is in fair agreement with that ($K_c=1.12\pm0.04$) obtained from Monte Carlo calculations (Tobochnik and Chester 1979) on (3.1), it should be stressed that it is rather subjective. Ultimately, (3.1) for all values of K iterates to the infinite-temperature fixed point $f\equiv 1$. Nevertheless, the Migdal approximation clearly distinguishes the low-temperature phase of the rotor model.

3.2. *The truncated model*

Turning to the truncated model (3.2), we observe that $f_t(\theta)$ is physical, i.e. $f_t(\theta)>0$ for all θ, if and only if $0\leqslant x\leqslant 1$. In this regime, $f_t(\theta)$ iterates rapidly to $f\equiv 1$ under (2.5), typical results (for $x=0.9$) being illustrated in figure 5(*a*). Allowing x to increase beyond unity does ultimately yield a 'fixed line' similar to that observed in the planar model (figure 5(*b*)). Using the same criterion for the onset of the massless phase as in the planar model we estimate $x_c\simeq 1.5$.

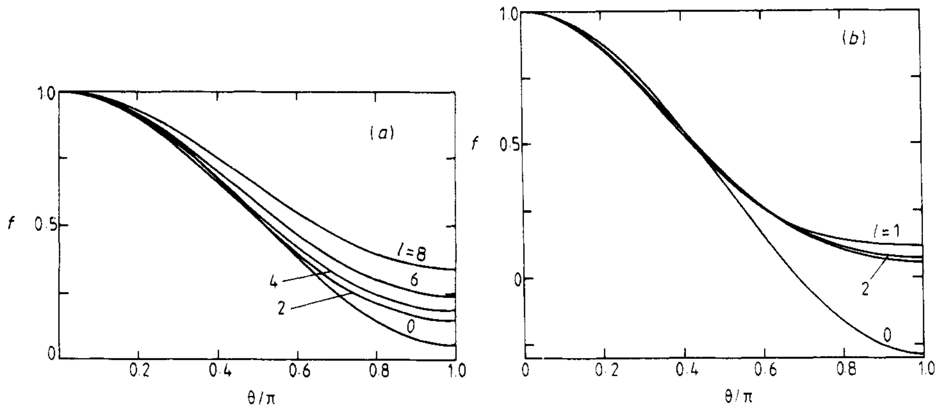

Figure 5. Successive Migdal iterations (labelled by l) of the truncated model for (a) $x = 0.9$ and (b) $x = 1.8$. Note in the latter case, the unphysical nature of $f_l(\theta)$ but the apparent convergence to a non-trivial 'fixed' function.

To my knowledge, no estimates of the critical coupling of the truncated model on the square lattice exist. The exact value (Domany *et al* 1981, Nienhuis 1982) for the hexagonal lattice corresponds in our notation to $x_c = \sqrt{2}$ which is also, as mentioned earlier, unphysical.

3.3. Step and line models

Our final results are for the step (3.3) and line (3.4) models. We consider first the step model with $\delta = \frac{1}{2}$ corresponding to the original definition of Guttmann *et al* (1972). One iteration of (3.3) with $\delta = \frac{1}{2}$ under (2.5) yields a line model with

$$\beta = (1 - e^{-4K})^2/(1 + e^{-8K}). \tag{3.5}$$

Note that for $0 < K < \infty$, $0 < \beta < 1$, in which range the line model is physical.

Iterating (2.5), beginning with a line model with $\beta < 1$ yields no 'fixed line', all values iterating rapidly to $f \equiv 1$. We conclude that the step model as formulated by Guttmann *et al* (1972) is always in a disordered high-temperature phase. The line model, on the other hand, does, like the truncated model, exhibit a fixed line for unphysical values of β, namely $\beta > \beta_c \simeq 3$.

The generalised step model can also be iterated once analytically under (2.5) to yield a modified line model

$$\bar{f}_s(\theta) = \begin{cases} 1 - \gamma^2|\theta|/2\pi(1 + 2\gamma\delta + \gamma^2\delta), & 0 \leq |\theta| < 2\delta\pi, \\ (1 + 2\gamma\delta)/(1 + 2\gamma\delta + \gamma^2\delta), & 2\delta\pi < |\theta| \leq \pi, \end{cases} \tag{3.6}$$

where $\bar{f}_s(\theta) = \bar{f}_s(\theta + 2\pi)$,

$$\gamma = e^{4K} - 1 \tag{3.7}$$

and $\delta \leq \frac{1}{2}$. (A similar result holds for $\delta > \frac{1}{2}$ but in this regime no fixed line is seen.) Subsequent iteration of (2.5) starting from (3.6) yields clear evidence of a 'fixed line' for $0.05 \leq \delta \leq 0.43$. The resulting phase diagram of the step model in the δ–K plane is shown in figure 6. For $\delta \geq \frac{1}{2}$, the step model is always disordered, while it appears that $K_c \to \infty$ as $\delta \to 0$ (see also the discussion at the end of § 4).

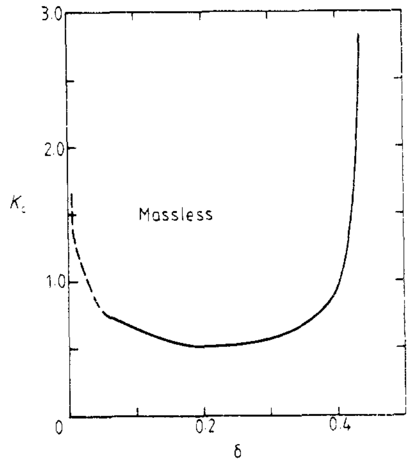

Figure 6. The phase diagram of the generalised step model in the δ–K plane. The full curve is the boundary determined by the Migdal recursion, while the broken curve is conjectural.

4. Discussion

The numerical results described in § 3 lead to the following conclusions.

(i) The planar rotor model has a massless low-temperature phase for $K > K_c \sim 1$.

(ii) The truncated model has a massless phase for $x > x_c \sim 1.5$, these values of x being, however, unphysical.

(iii) The original step model (Guttmann *et al* 1972) is for all K in a disordered phase.

(iv) The modified step model exhibits a massless phase for all $\delta < \delta_c \lesssim \frac{1}{2}$.

(v) In the massless region all models iterate to a 'fixed function' that can be well represented by a Villain model.

The key question is to what extent these conclusions are valid beyond the Migdal approximation.

For the planar model the existence of a low-temperature massless phase is well established (Kosterlitz 1974, José *et al* 1977, Hamer and Barber 1981), while the conclusion regarding the step model with $\delta = \frac{1}{2}$ is consistent with the latest series work (Guttmann and Nymeyer 1978). It would be interesting to explore the generalised step models by other techniques. As discussed in § 3.2, the truncated model has not previously been explored on the square lattice, but our results support the assumption of Nienhius (1982) that this model is a faithful representation of the O(2)-symmetric universal class. Moreover, the fact that at criticality the model is unphysical does not appear to be very significant†.

It is informative to close by considering these conclusions in the light of the Kosterlitz–Thouless (1973) criterion for the stability of the ordered phase to creation of free vortices. This criterion is based on an estimate of the incremental free energy Δf associated with a spin configuration (vortex) of the form shown in figure 7.

† The conclusion should be regarded with some care, since the Migdal transformation (2.5) *automatically* produces a physical $\bar{f}(\theta)$ on one iteration.

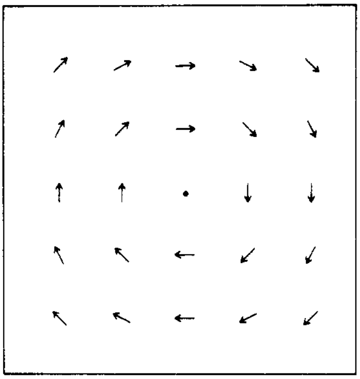

Figure 7. Spin configuration of an isolated vortex of an O(2) Hamiltonian in two dimensions.

Explicitly, for a vortex of radius R in a system of size R

$$\Delta f / k_{\mathrm{B}} T \simeq E(R) - 2 \ln(R/a), \tag{4.1}$$

where the two terms represent the energy and entropy of the vortex respectively. Approximating the lattice by a continuum yields

$$E(R) \simeq -\int_0^R 2\pi r V(\Delta\theta_r)\, \mathrm{d}r \tag{4.2}$$

with $\Delta\theta_r \sim 1/r$ and $V(\theta)$ the interaction energy as defined in (2.1). If $V(\theta) \sim -\frac{1}{2}P\theta^2$ as $\theta \to 0$ with $P = -V''(0)$, $E(R)$ for large R behaves as

$$E(R) \sim \pi P \ln(R/a), \tag{4.3}$$

and thus dominates the second (entropy) term in (4.1) for $\pi P > 2$, implying the stability of the system to the creation of a free vortex. The Kosterlitz–Thouless estimate of critical value of P for the onset of the massless phase is thus

$$P_{\mathrm{c}} = -V_{\mathrm{c}}''(0) = 2/\pi. \tag{4.4}$$

Applying this criterion to the planar model yields $K_{\mathrm{c}} = 2/\pi \simeq 0.64$, while for the truncated model we obtain $x_{\mathrm{c}} = 2/(\pi - 2) \sim 1.75$. Both these estimates differ significantly from the actual values but it is significant that this simple physical criterion also predicts that the truncated model is always disordered for physical values of x.

The criterion (4.4) cannot be applied to the line model since $V_l(\theta)$ is not differentiable at $\theta = 0$, while for the step model with $\delta \geqslant \frac{1}{2}$, as observed by Guttmann and Nymeyer (1978), $E(R) \equiv 0$; there being no energy cost associated with a spin configuration of the form shown in figure 7. This suggests that free vortices are easily created and disorder the system in agreement with the numerical results. However, for $\delta < \frac{1}{2}$, the energy of a vortex is still only O(1) in R, *not* logarithmic. The only effect of decreasing δ is to give an energy cost to the core of the vortex; all spins beyond a radius $r \gtrsim 1/\pi\delta$ can be rotated back to the fully aligned state without any cost in energy. Such core effects are not expected to affect the critical behaviour.

Are our results for the step model with $\delta < \frac{1}{2}$ thus an artifact of the Migdal approximation? This is possible. On the other hand, if we accept the faithfulness of the approximation, it is perhaps significant that whilst the Kosterlitz–Thouless criterion

(4.4) cannot be applied to either the step model or its first iterate (the line model), it can be applied to the second iterate. The resulting phase diagram in the δ–K plane is similar to that found within the Migdal approximation (figure 6) but shifted to lower values of K. Whether this is a reasonable representation of the phase diagram of this model and whether the transition, if it exists, is a Kosterlitz–Thouless transition require other techniques.

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